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THE METHOD OF NON-LINEAR TIME TRANSFORMATION IN BOUNDARY-VALUE PROBLEMS OF POTENTIAL THEORY WITH MOVING BOUNDARIES FOR THE NON-LINEAR WAVE EQUATION†

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A new method for solving boundary-value problems for the wave equation [1–3] with moving boundaries is used to obtain a solution of a boundary-value problem with boundary conditions of three types [4].

THE MOST general method for solving problems of heat conduction, diffusion and wave processes with moving boundaries uses expansions in terms of the instantaneous natural frequencies [5]. In practical work, however, this method is very laborious.

1. MATHEMATICAL FORMULATION OF THE BOUNDARY-VALUE PROBLEM

We consider [5] the non-steady-state boundary-value problem of axially symmetric waves radiating from a moving surface, where they are generated by physical processes of some kind. It is required to determine a potential function φ that satisfies the linear wave equation under conditions of symmetry with respect to the space coordinate r:

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{m}{r}\frac{\partial\varphi}{\partial r}-\frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2}=0 \tag{1.1}$$

where $m = 0, 1, 2$, t is the time and c_0 is the velocity of propagation of the perturbations. The initial data in the domain $r \geqslant R_0$ are assumed to be zero:

$$\varphi\,|_{t=0}=\partial\varphi/\partial t\,|_{t=0}=0 \tag{1.2}$$

In addition, we will assume that the unknown φ satisfies a radiation condition, according to which the solution of Eq. (1.1) will contain only travelling waves. The solution of Eq. (1.1) with initial data (1.2) and the radiation condition is

$$\varphi(r, t)=f(t^\circ)\,r^{-m/2},\quad t^\circ=t-(r-R_0)/c_0 \tag{1.3}$$

where t° is the wave argument. Note that when $m = 1$ this solution is approximate, and valid [6] for fairly large values of r. A more careful analysis yields a condition for solution (1.3) to be applicable when $m = 1$: $c_0t/R_0 \ll 1$.

The form of the function f in (1.3) is determined by the boundary condition at the moving boundary, which may be of one of three types [4]:

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$$\varphi\,|_{r=R(t)} = q_1(t) \quad \text{(type 1)} \tag{1.4}$$

$$\partial\varphi/\partial r\,|_{r=R(t)} = q_2(t) \quad \text{(type 2)} \tag{1.5}$$

$$(\partial\varphi/\partial r + \alpha R_0^{-1}\varphi)\,|_{r=R(t)} = q_3(t) \quad \text{(type 3)} \tag{1.6}$$

where α is an arbitrary constant and $R_0 \neq 0$.

In boundary conditions of these three types the law of motion of the boundary is expressed by an arbitrary but continuous function of time. The function f may be interpreted as the intensity of some *ad hoc* point source placed at the origin, outside the domain of definition of φ for the boundary-value problem. The solution of problem (1.1)–(1.6) will be sought in the domain $t \geqslant 0$, $r \geqslant R_0$. Each of the three types of boundary condition (1.4)–(1.6) will be considered separately.

2. BOUNDARY CONDITION OF THE FIRST TYPE

Solving (1.3) with condition (1.4), we obtain

$$f(t - (R(t) - R_0)/c_0) = R^{m/2}(t)\, q_1(t) \tag{2.1}$$

We now use the transformation [1–3]

$$\xi = t - (R(t) - R_0)/c_0 \tag{2.2}$$

Let $t = w(\xi)$ be the solution of Eq. (2.2). Then the use of this transformation enables us to express the solution in the form

$$f(\xi) = R^{m/2}(w(\xi)) \tag{2.3}$$

The variable ξ in (2.2) and (2.3) is actually a new time variable. Therefore, setting $\xi = t^\circ$ in (2.3) and substituting the result into (1.3), we obtain the required solution of the boundary-value problem of the first type:

$$\varphi(r, t) = [R(w(t^\circ))/r]^{m/2}\, q_1(w(t^\circ)) \tag{2.4}$$

Indeed, the function (2.4) is a solution of the wave equation, because it corresponds to the form of (1.3) and is a function of the wave argument. On the other hand, the construction of f was determined from the boundary condition (1.4) specified on the moving boundary.

Consider the special case of a boundary moving at a constant velocity:

$$R(t) = R_0 + v_0 t, \quad v_0 = \text{const}, \quad w(t^\circ) = t^\circ/(1 - M_0)$$
$$M_0 = v_0/c_0, \quad R(w(t^\circ)) = R_0 + v_0 t^\circ/(1 - M_0)$$

Formula (2.4) then becomes

$$\varphi(r, t) = \left[\frac{R_0}{r}\left(1 + \frac{v_0 t^\circ}{R_0(1 - M_0)}\right)\right]^{m/2} q_1\left(\frac{t^\circ}{1 - M_0}\right) \tag{2.5}$$

3. BOUNDARY CONDITION OF THE SECOND TYPE

Solving (1.3) with condition (1.5), we obtain an equation for the function $f(\xi)$:

$$\frac{df(\xi)}{d\xi} + \frac{c_0 m}{2}\,\frac{f(\xi)}{R(w(\xi))} = -c_0 R^{m/2}(w(\xi))\, q_2(w(\xi)) \tag{3.1}$$

where $t = w(\xi)$ is the solution of Eq. (2.2).

The solution of Eq. (3.1) is known; it is

$$f(\xi) = -c_c \begin{cases} \int_0^{\xi} q_2(w(\xi_1))\, d\xi_1, & m = 0 \\ F(\xi) \int_0^{\xi} \frac{R^{m/2}(w(\xi_1))\, q_2(w(\xi_1))}{F(\xi_1)}\, d\xi_1, & m = 1, 2 \end{cases} \tag{3.2}$$

$$F(\xi) = \exp\left[-\frac{c_0 m}{2} \int_0^{\xi} \frac{d\xi}{R(w(\xi))}\right]$$

Substituting (3.2) into (1.3) and replacing the argument ξ by the wave argument t°, we obtain the required solution:

$$\varphi(r, t) = -c_0 \begin{cases} \int_0^{w(t^\circ)} \left[1 - \frac{1}{c_0}\frac{dR}{dt}\right] q_2(t)\, dt, & m = 0 \\ \left[\frac{R(w(t^\circ))}{r}\right]^{m/2} G(w(t^\circ)) \int_0^{w(t^\circ)} \frac{q_2(t)}{G(t)} \left[1 - \frac{1}{c_0}\frac{dR}{dt}\right] dt, & m = 1, 2 \end{cases}$$

$$G(t) = \exp\left[-\frac{c_0 m}{2} \int_0^{t} \frac{dt}{R(t)}\right] \tag{3.3}$$

In the special case of a boundary moving at a constant velocity, solution (3.3) becomes

$$\varphi(r, t) = -c_0(1 - M_0) \times$$

$$\times \begin{cases} \int_0^{t^\circ/(1-M_0)} q_2(t)\, dt, & m = 0 \\ \left(\frac{R_0}{r}\right)^{m/2} \psi^{m/2(1-1/M_0)}(t^\circ) \int_0^{t^\circ/(1-M_0)} q_2(t)\, \psi^{m/(2M_0)}(t)\, dt, & m = 1, 2 \end{cases}$$

$$\psi(t) = 1 + \frac{v_0 t}{R_0(1 - M_0)}\,.$$

4. BOUNDARY CONDITION OF THE THIRD TYPE

Solving (1.3) with condition (1.6) we obtain

$$\frac{df(\xi)}{d\xi} + \frac{mc_0}{2}\left[\frac{1}{R(w(\xi))} - \frac{2\alpha}{mR_0}\right] f(\xi) = -c_0 R^{m/2}(w(\xi))\, q_2(w((\xi)) \tag{4.1}$$

Using the solution of Eq. (4.1) and noting (1.3), we find the required solution:

$$
\varphi(r,t) = -c_0 \begin{cases} \exp\left(\dfrac{c_0\alpha}{R_0} w(t^\circ)\right) \displaystyle\int_0^{w(t^\circ)} q_3(t) \exp\left(-\dfrac{c_0\alpha}{R_0} t\right) dt, & m = 0 \\ \left[\dfrac{R(w(t^\circ))}{r}\right]^{m/2} H(w(t^\circ)) \displaystyle\int_0^{w(t^\circ)} \dfrac{q_3(t)}{H(t)} \left[1 - \dfrac{1}{c_0}\dfrac{dR}{dt}\right] dt, & m = 1, 2 \end{cases} \tag{4.2}
$$

$$
H(t) = \exp\left[-\frac{c_0 m}{2} \int_0^t \frac{dt_1}{R(t_1)} + \frac{\alpha}{c_0} t\right]
$$

For a boundary moving at a constant velocity the solution (4.2) gives

$$
\varphi(r,t) = -c_0 \begin{cases} \displaystyle\int_0^{t^\circ/(1-M_0)} q_3(t) \exp\left(-\dfrac{\alpha c_0}{R_0} t\right) dt, & m = 0 \\ (1 - M_0)\left(\dfrac{R_0}{r}\right)^{m/2} \psi^{m/4(1/2 - 1/M_0)} \psi(t^\circ) \exp\left(\dfrac{\alpha t^\circ}{R_0(1 - M_0)}\right) \times \\ \times \displaystyle\int_0^{t^\circ/(1-M_0)} q_3(t) \psi^{m/(2M_0)} \psi(t) \exp\left(-\dfrac{\alpha t}{R_0}\right) dt, & m = 1, 2 \end{cases} \tag{4.3}
$$

5. SOLUTION OF THE DEFINING EQUATION

The main difficulty encountered when solving a boundary-value problem with moving boundaries by the non-linear transformation method is to determine the solution of the defining algebraic equation (2.2). We have already found an analytical solution for boundaries moving at a constant velocity v_0. In the more-general case when the law of motion has the form $R(t) = R_0 + v_0 t + a_0 t^2/2$, we obtain a quadratic equation for t, whose solution is

$$
w(\xi) = (1 - M_0)\frac{c_0}{a_0}\left\{1 - \left[1 - 2\frac{a_0(R_0/c_0 - \xi)}{c_0(1 - M_0)^2}\right]^{1/2}\right\}
$$

In the case of an arbitrary law of motion, but subject to the condition $[R_1/(c_0 t)]^2 \ll 1$, the solution of Eq. (2.2) may be determined by successive approximations; the first three approximations are:

$$
w_1(\xi) = \xi, \quad w_2(\xi) = \xi + \frac{R_1(\xi)}{c_0}, \quad w_3(\xi) = \xi + \left(1 + \frac{1}{c_0}\frac{dR_1}{d\xi}\right)\frac{R_1(\xi)}{c_0}
$$

where $R_1(\xi) = R(\xi) - R_0$.

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A SOLUTION OF AN INVERSE PROBLEM FOR THE WAVE EQUATION WITH NON-LINEAR CONDITIONS IN DOMAINS WITH MOVING BOUNDARIES†

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A new approach is proposed for solving problems with moving boundaries. Assuming spherical symmetry, wave phenomena are considered in the case of a surface, of arbitrary initial radius, moving in a compressible medium at a velocity governed by an arbitrary law. Formulas suitable for solving both the inverse and direct problems are obtained.

ATTEMPTS to allow for the mobility of the boundaries in wave-equation situations have hitherto been confined mainly to cases in which the boundary conditions are satisfied on the moving boundaries (the direct problem) [1, 2]. The method used in [1] reduces such situations to an infinite system of first-order linear differential equations. In the case considered below an additional condition is specified not at the moving boundary but at a fixed point of the wave zone (the inverse problem), and the problem is to determine the functions of interest at other points, including the moving boundaries. This is to be done without knowledge of the law governing the variation of the boundaries, which is also to be determined. In addition, the additional condition may even be non-linear.

The essence of the approach is to determine the relationship between the values of the unknown functions at the moving boundaries and at other points taking into account the actual delays [3]. In some cases, such as a moving cylindrical surface [4] or a penetrable spherical boundary [5], explicit formulas can be derived for the functions.

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